Topology and its Applications 35 (1990) 53-69
North-Holland

A TOPOLOGICAL COMPLETENESS CONCEPT WITH APPLICATIONS TO THE OPEN MAPPING THEOREM AND THE SEPARATION OF CONVEX SETS

Dominikus NOLL
Mathematisches Institut B, Pfaffenwaldring, 7000 Stuttgart 80, FRG

Received 6 May 1988
Revised 9 September 1988

We introduce a completeness concept for convex sets in locally convex vector spaces which is based on the topological notion of p-completeness (also weak α-favourability). Using purely topological methods, we then establish an open mapping theorem for convex multifunctions and a separation theorem for convex sets generalizing the Tuckey-Klee separation theorem. Finally, we indicate that our notion of completeness encompasses Jameson's CS-closedness for convex sets, which hereby is shown to be essentially a topological notion.

AMS (MOS) Subj. Class: 02G20, 54C10, 46A30, 52A05

CS-closedness	separation of convex sets
feebly open map	open mapping theorem
p-completeness	pseudo-completeness

Introduction

In this paper we are concerned with a topological notion of completeness, called *p-completeness*, which in [9, 10] was shown to bear a close relation to the open mapping and closed graph theorems in topological spaces. The concept of p-completeness, which lies between the stronger Čech completeness and the weaker Baire category, was already discussed by White [16] under the name "*weak α-favourability*", referring to the Banach-Mazur game characterization given in that paper. Our present investigation deals with several applications of p-completeness in topology and in functional analysis.

In particular, the following results are derived. In Section 1 we start with a purely topological result of fundamental relevance to the whole paper. We prove that a continuous dense and *nearly feebly open* mapping f (i.e., $\operatorname{int} \overline{f(V)} \neq \emptyset$ for nonempty open V) from a p-complete space E to a metrizable space F maps residual subsets of E onto residual subsets of F. As a consequence we obtain results concerning the openness of a continuous nearly feebly open bijection $f: E \to F$ at the points of a generic subset of E.

In Section 2 we discuss the concept of *pseudo-completeness* for convex cones and convex sets in locally convex vector spaces. Here a convex cone C with vertex 0 in a locally convex vector space E (i.e., $C+C\subset C, \mathbb{R}_+C\subset C$) is called pseudo-complete if it is a p-complete topological space when endowed with a special topology σ, called the *cone topology*, which differs from the original (locally convex) topology on C. This topology has been introduced by Saint-Raymond [13] in a special situation and has been further investigated in [11]. It turns out that the class of pseudo-complete convex sets in a locally convex Fréchet space is fairly large. In particular it contains all *CS-closed* sets in the sense of Jameson [4], and therefore, as a consequence of a result of Fremlin and Talagrand [3] the class of all convex G_δ-sets.

The concept of pseudo-completeness for convex cones and sets permits us to apply purely topological methods in functional analysis. As a first application we obtain in Section 4 an open mapping theorem for multifunctions Φ between Banach spaces E, F having pseudo-complete convex graph $G(\Phi)$ in $E\times F$. This result contains as special cases the corresponding open mapping theorem for closed graph convex multifunctions obtained by Robinson [12] as well as the open mapping theorem for CS-closed graph functions from [4]. In Section 5 we present as a second application a separation theorem for pseudo-complete convex sets generalizing the Tuckey–Klee separation theorem (see [6, 14]).

In the final Section 6 we clarify somewhat the relation between Jameson's CS-closedness and our notion of pseudo-completeness. We introduce an infinite two-person game Γ between players I, II on a convex cone C with vertex 0 in a locally convex Fréchet space. It turns out that then C is pseudo-complete precisely when player II has a winning strategy in this game Γ. On the other hand, the cone C is CS-closed when, roughly speaking, *every* reasonable strategy for player II is automatically winning, so that player II actually runs into difficulties only when he decides to lose the game.

Preliminaries

Our terminology concerning notions from general topology is based on the book [2]. Functional analytic concepts are covered by [7] or [5]. In the following, we briefly list some notions of special interest in this paper.

0.1. *Webs*

Let E be a topological space. A pair (φ, T) consisting of a tree $T=(T, \leqslant_T)$ of height $\aleph_0$ (cf. [8, p. 84]) and a mapping φ from T to the topology of E is called a web on E if the following conditions are fulfilled:

(i) The set $\{\varphi(t)\colon t\in T\}$ is a pseudo-base for E (i.e., every nonempty open U in E contains some nonempty $\varphi(t)$);

(ii) for fixed $t\in T$ the set $\{\varphi(s)\colon t<_T s\in T\}$ is a pseudo-base for the subspace $\varphi(t)$.

If in conditions (i) and (ii) the term "pseudo-base" is replaced by "base", then the web (φ, T) is called *strict.*

0.2. *p-completeness*

A web (φ, T) on a topological space E is called p-complete if it fulfills the following condition:

$$\text{Whenever } (t_n) \text{ is a cofinal branch in } T \text{ (i.e., } t_n <_T t_{n+1} \text{ for all } n\text{) such that } \varphi(t_n) \neq \emptyset \text{ holds for all } n, \text{ then the set } \bigcap \{\varphi(t_n) : n \in \mathbb{N}\} \text{ is nonempty.} \tag{p}$$

A topological space is called p-complete respectively strictly p-complete if it admits a p-complete respectively strictly p-complete web. Every Čech complete space is strictly p-complete and hence p-complete. p-complete spaces are Baire spaces. The class of (strictly) p-complete spaces is closed under continuous open images and under products. Strict p-completeness is G_δ-hereditary and p-completeness is inherited by open subspaces and by dense G_δ-subspaces. The p-complete spaces are known under the name "weakly α-favourable spaces" introduced in [16]. A similar game-theoretic characterization may be given for the class of strictly p-complete spaces when the strong game of Choquet (see [1, 16]) is used.

0.3. *Feebly openness*

A mapping f from a topological space E to a topological space F is called *feebly open* if $\operatorname{int} f(V) \neq \emptyset$ in F for every nonempty open V in E. In analogy with the notion of nearly openness we may now introduce the concept of nearly feebly openness. The mapping $f: E \to F$ is called *nearly feebly open* if $\operatorname{int} \overline{f(V)} \neq \emptyset$ holds in F for every nonempty open V in E.

1. Preserving residual subsets

In this section we establish a topological result of basic nature being of importance for the considerations to follow. We start with:

Proposition 1.1. *Let E be a p-complete topological space and let F be a regular space. Let $f: E \to F$ be a continuous dense and nearly feebly open mapping. Then F is p-complete.*

Proof. Let (φ, T) be a p-complete web on E. We may assume that the sets $\varphi(t)$ have the following additional property: $f(\varphi(t)) \subset \operatorname{int} \overline{f(\varphi(t))}$. Indeed, this is a consequence of the fact that the nonempty open sets V in E having $f(V) \subset \operatorname{int} \overline{f(V)}$ form a pseudo-base in E. For let U in E be nonempty and open. Then letting

$$V = U \cap f^{-1}(\operatorname{int} \overline{f(U)}),$$

we obtain a nonempty open subset V of U with $f(V) \subset \operatorname{int} \overline{f(V)}$.

Let us define a mapping ψ on T by setting

$$\psi(t) = \operatorname{int}\overline{f(\varphi(t))},$$

$t \in T$. This actually defines a web on F. We check condition (i) of Section 0.1. Let W be a nonempty open subset of F. By regularity choose a nonempty open subset U of W having $\bar{U} \subset W$. Now $f^{-1}(U)$ is a nonempty open subset of E in view of the fact that f is dense. Therefore we find a nonempty $\varphi(t)$ contained in $f^{-1}(U)$, and this implies $\psi(t) \subset W$. By a similar argument one checks the validity of condition (ii).

Let us now prove that (ψ, T) fulfills condition (p) from Section 0.2. So let (t_n) in T be given with $t_n <_T t_{n+1}$ and $\psi(t_n) \neq \emptyset$ for all n. This implies $\varphi(t_n) \neq \emptyset$ for all n, hence by condition (p) for (φ, T) we find some $x \in \varphi(t_n)$, $n \in \mathbb{N}$. This gives $f(x) \in f(\varphi(t_n)) \subset \psi(t_n)$ for every n. Hence F is p-complete with (ψ, T). □

Proposition 1.1 has the following consequence.

Theorem 1.2. *Let E be a p-complete topological space and let F be a metrizable space. Let $f: E \to F$ be a continuous dense and nearly feebly open function. Then f maps residual subsets of E onto residual subsets of F.*

Proof. Let G be a dense G_δ-subset of E. Then G is a p-complete space since p-completeness is inherited to dense G_δ-subspaces. Since f is nearly feebly open and dense, the same is true for $f|G: G \to F$, so by Proposition 1.1 the space $f(G)$ is p-complete. It remains to prove that every metrizable p-complete space contains a dense completely metrizable subspace, for then it follows from the theorem of Alexandrov that $f(G)$ contains a dense subspace which is a G_δ in F, so $f(G)$ is residual in F. Hence we are led to prove:

Proposition 1.3 (cf. [16, Theorem 3(11)]). *Every metrizable p-complete space contains a dense completely metrizable subspace.*

Proof. Let E be metrizable and p-complete with (φ, T). We may assume that (φ, T) has the following additional properties:

(i′) The set $\{\varphi(t): t \in T_0\}$ is a pseudo-base for E;

(ii′) for fixed $t \in T_n$ the set $\{\varphi(s): t <_T s \in T_{n+1}\}$ is a pseudo-base for $\varphi(t)$;

(iii) for $t \in T_n$, $\varphi(t)$ has diameter $\leqslant 2^{-n}$ (with respect to some fixed metric for E).

Here T_n denotes the set of $t \in T$ having height n in the tree T.

Using transfinite induction, we may now define a mapping ψ on T_0 such that either $\psi(t) = \varphi(t)$ or $\psi(t) = \emptyset$ and where $\bigcup\{\psi(t): t \in T_0\}$ is dense in E and $t, t' \in T_0$, $t \neq t'$ implies $\psi(t) \cap \psi(t') = \emptyset$. Now let $t \in T_0$ be fixed. Define $\psi(s)$ for the immediate successors s of t in T using transfinite induction such that either $\psi(s) = \varphi(s)$ or $\psi(s) = \emptyset$ and where $\bigcup\{\psi(s): t <_T s \in T_1\}$ is dense in $\psi(t)$ and such that $t <_T s \in T_1$, $t <_T s' \in T_1$ and $s \neq s'$ implies $\psi(s) \cap \psi(s') = \emptyset$. This defines ψ on the level T_1.

Continuing in this way, we obtain a web (ψ, T) on E which is disjoint in the sense that $t, t' \in T_n$, $t \neq t'$ implies $\psi(t) \cap \psi(t') = \emptyset$. Now let

$$G_n = \bigcup \{\psi(t): t \in T_n\};$$

then G_n is open dense in E and, E being a Baire space, the set $G = \bigcap \{G_n: n \in \mathbb{N}\}$ is a dense G_δ in E. We claim that G is completely metrizable. Indeed, this follows by setting

$$\chi(t) = G \cap \psi(t),$$

$t \in T$, for now (χ, T) is a strict p-complete web on G which by construction of ψ is disjoint. This shows that G is a strongly zero-dimensional completely metrizable space. □

The proof of Theorem 1.2 being complete, we may now derive the following open mapping theorem which is closely related with our open mapping theorem [9, Theorem 3].

Theorem 1.4. *Let E, F be metrizable topological spaces and suppose E contains a dense completely metrizable subspace. Let $f: E \to F$ be a continuous and nearly feebly open bijection. Then there exists a dense G_δ-subset G of E such that f is open at every $x \in G$, i.e., $f(x) \in \operatorname{int} f(U)$ whenever U is a neighbourhood of x in E.*

Proof. (1) First we prove that there exists a dense G_δ-subset G of E such that f is nearly open at every $x \in G$, which means that $f(x) \in \operatorname{int} \overline{f(U)}$ for every neighbourhood U of x in E. Indeed, recall that the nonempty open sets V in E having $f(V) \subset \operatorname{int} \overline{f(V)}$ form a subbase for E. Now let G_n be the union of all sets of this kind having diameter $\leqslant 1/n$ with respect to some fixed metric for E. Then $G = \bigcap \{G_n: n \in \mathbb{N}\}$ is as desired.

(2) Let V, W be nonempty open sets in E having $f(V) \subset \operatorname{int} \overline{f(V)}$, $f(W) \subset \operatorname{int} \overline{f(W)}$, and suppose $\operatorname{int} \overline{f(V)} \cap \operatorname{int} \overline{f(W)} \neq \emptyset$. We claim that this implies $V \cap W \neq \emptyset$. Indeed, let $O = \operatorname{int} \overline{f(V)} \cap \operatorname{int} \overline{f(W)}$, and let $V_1 = V \cap f^{-1}(O)$, $W_1 = W \cap f^{-1}(O)$. Then we have $\overline{f(V_1)} = \overline{f(W_1)} = \bar{O}$. By Theorem 1.2 the sets $f(V_1)$ and $f(W_1)$ are both residual in O, since $f | V_1$ and $f | W_1$ are nearly feebly open and V_1, W_1 are open in E and hence p-complete. Since O is a Baire space, we derive $f(V_1) \cap f(W_1) \neq \emptyset$, and f being injective, this implies $V_1 \cap W_1 \neq \emptyset$, hence $V \cap W \neq \emptyset$.

(3) Let $x \in G$ be fixed. We prove that f is open at x. Let U be an open neighbourhood of x. Choose an open neighbourhood V of x such that $\bar{V} \subset U$. We prove $\operatorname{int} \overline{f(V)} \subset f(U)$, which clearly implies $x \in \operatorname{int} f(U)$ by the definition of G. Let $z \in \operatorname{int} \overline{f(V)}$, $z = f(y)$. It suffices to show $y \in \bar{V}$. Let W be an open neighbourhood of y. We have to prove $V \cap W \neq \emptyset$. By continuity of f we may assume that $f(W) \subset \operatorname{int} \overline{f(V)}$, so $\operatorname{int} \overline{f(W)} \cap \operatorname{int} \overline{f(V)} \neq \emptyset$. By (2) this gives $V \cap W \neq \emptyset$, hence the proof is complete. □

Remark 1.5. Metrizability of the space E is essential in Theorem 1.4. For let $F=\mathbb{R}$ with the Euclidean topology and let E be the set $\mathbb{R}$ endowed with the so-called Sorgenfrey topology, i.e., the topology generated by the intervals $[a, b)$, then E is p-complete, $\mathrm{id}: E\to F$ is continuous and nearly feebly open, but clearly id is not open at any $x\in E$.

Corollary 1.6. *Let E, F be completely metrizable topological spaces and let $f: E\to F$ be a mapping whose graph $G(f)$ is a G_δ-set in $E\times F$. Suppose that for every open set W in F we have*

$$f^{-1}(W)\subset\overline{\operatorname{int}\overline{f^{-1}(W)}}.$$

Then there exists a dense G_δ-subset G of $G(f)$ such that for every $(x,f(x))\in G$, f is continuous at x.

Proof. Consider the mapping $g: G(f)\to E$, $(x,f(x))\to x$. Then g is a continuous bijection from a completely metrizable space to a metrizable space. It suffices to prove that g is nearly feebly open, for then Theorem 1.4 provides a dense G_δ-subset G of $G(f)$ at the points of which g is open, and the latter clearly means that f is continuous at these points.

Let $(x,f(x))$ be fixed and let $W=(U\times V)\cap G(f)$ be a neighbourhood of $(x,f(x))$ in $G(f)$, where U is an open neighbourhood of x in E and V is an open neighbourhood of $f(x)$ in F. It suffices to show that $\operatorname{int}\overline{g(W)}\neq\emptyset$, for W is a typical open set in $G(f)$. But notice that $g(W)=U\cap f^{-1}(V)$. By assumption the interior of $\overline{f^{-1}(V)}$ is dense in $f^{-1}(V)$, hence $U\cap\operatorname{int}\overline{f^{-1}(V)}$ must be nonempty. Clearly this implies $\operatorname{int}\overline{g(W)}\neq\emptyset$ as desired. $\square$

Remark 1.7. A mapping $f: E\to F$ is called nearly feebly continuous if for every nonempty relatively open subset V of $f(E)$ the set $\overline{f^{-1}(V)}$ has nonempty interior in E. Note that this is a slightly weaker property than the one claimed in Corollary 1.6 above. It can be shown, however, that nearly feebly continuity is not sufficient to obtain the conclusion of Corollary 1.6.

2. Pseudo-complete sets

Let E be a separated real locally convex vector space and let C be a convex cone with vertex 0 in E (i.e., $C+C\subset C$, $\mathbb{R}_+C\subset C$). We denote by τ the trace of the topology of E on C. Notice that τ is not invariant under the translations $x\to x+y$, $y\in C$, preserving C, i.e., for a τ-neighbourhood V of some $x\in C$, $V+y$ is not a τ-neighbourhood of $x+y$. We therefore introduce a new topology σ on C, called the cone topology, which *has* this property. We choose as a base of neighbourhoods of $x\in C$ with respect to σ the sets

$$x+(U\cap C),$$

where U varies over the neighbourhoods of 0 in E. This defines a topology finer than τ, which is invariant under translations $x \to x+y$, $y \in C$, in the sense that any such translation maps the space (C, σ) homeomorphically onto its open subspace $(C+y, \sigma)$. The topology σ has first been considered in [13] in a special situation and has been further investigated in [11].

Let us consider an instructive example. Let $E = \mathbb{R}$, $C = \mathbb{R}_+$. Then τ is the Euclidean topology on C, while σ is the Sorgenfrey topology on C.

Definition 2.1. A convex cone C with vertex 0 in a separated locally convex vector space E is called *(strictly) pseudo-complete* if C is a (strictly) p-complete topological space in its cone topology σ.

The following result shows that the class of pseudo-complete cones in a locally convex Fréchet space is fairly large. It tells that all CS-closed cones in the sense of Jameson [4] are pseudo-complete. Recall that a convex set C in a locally convex vector space E is called *CS-closed* if every convergent series $\sum_{n=1}^{\infty} \lambda_n x_n$ with $0 \leq \lambda_n \leq 1$, $\sum_{n=1}^{\infty} \lambda_n = 1$, $x_n \in C$ actually converges to an element of C.

Proposition 2.2. *Every CS-closed convex cone with vertex* 0 *in a locally convex Fréchet space E is strictly pseudo-complete. In particular, every convex G_δ-cone in E is strictly pseudo-complete.*

We postpone the proof of the first part of Proposition 2.2 until Section 6. The second part of the statement is a consequence of the first part and a result of Fremlin and Talagrand [3] stating that convex G_δ-sets in a Fréchet space are CS-closed.

We wish to extend the notion of pseudo-completeness to arbitrary convex sets. This is done by making use of the following auxiliary construction. Given a convex set C in a separated locally convex vector space E, we denote by $\tilde{C}$ the convex cone with vertex $(0, 0)$ in $E \times \mathbb{R}$ generated by the set $C \times \{1\}$, i.e., $\tilde{C} = \mathbb{R}_+(C \times \{1\})$.

Definition 2.3. A convex set C in a separated locally convex vector space E is called *(strictly) pseudo-complete* if the cone $\tilde{C}$ associated with C in $E \times \mathbb{R}$ is (strictly) pseudo-complete in the sense of Definition 2.1.

Remark 2.4. Notice that Definition 2.3 gives sense also in the case where C is already a convex cone with vertex 0 in E. Indeed, in this case we have $\tilde{C} = C \times \mathbb{R}_+$ and hence (strict) pseudo-completeness of C in the sense of Definition 2.1 is equivalent to the (strict) pseudo-completeness of $\tilde{C}$ in the sense of Definition 2.1. This may be seen by observing that the product of the cone topologies on C and $\mathbb{R}_+$ is just the cone topology on $\tilde{C}$, and by taking into account that the cone topology on $\mathbb{R}_+$ is strictly p-complete.

Proposition 2.5. *Every CS-closed convex set in a locally convex Fréchet space E is strictly pseudo-complete. In particular, every convex G_δ-set in E is strictly pseudo-complete.*

Proof. This follows from Proposition 2.2 by observing that the cone $\tilde{C}$ associated with a CS-closed convex set C is itself CS-closed. □

Example 2.6. We give an example of a CS-closed hence strictly pseudo-complete cone in a Banach space which is of the first category in itself with respect to the topology induced by the norm. Let $E = l^1(\mathbb{N})$ be the Banach space of absolutely summable sequences and let C be the order-cone of the lexicographic ordering on E, i.e.,

$$C = \{x \in l^1(\mathbb{N}) \colon x(1) = \cdots = x(n-1) = 0,\ x(n) \neq 0 \Rightarrow x(n) > 0\}.$$

It is easy to see that C is CS-closed. But C may be represented as $C = \bigcup \{C_{n,m} \colon n, m \in \mathbb{N}\}$, where

$$C_{n,m} = \left\{x \in C \colon x(1) = \cdots = x(n-1) = 0,\ x(n) \geq \frac{1}{m}\right\},$$

and these sets are closed but have no interior points relative to C.

3. Semi-closed sets

Jameson [4] calls a convex set C in a locally convex vector space E *semi-closed* if $\operatorname{int} C = \operatorname{int} \bar{C}$ holds. He proves that CS-closed sets are semi-closed. Here we obtain the following more general:

Theorem 3.1. *Every pseudo-complete convex set C in a locally convex Fréchet space E is semi-closed.*

Proof. In case $\operatorname{int} \bar{C} = \emptyset$ there is nothing to prove. So let $x \in \operatorname{int} \bar{C}$. We have to prove $x \in C$. Now observe that pseudo-completeness is invariant under translations. So we may assume that x is the origin in E.

Let $\tilde{C}$ be the cone associated with C in $E \times \mathbb{R}$. Let K denote the convex cone with vertex $(0, 0)$ in $E \times \mathbb{R}$ defined by

$$K = \tilde{C} - (\{0\} \times \mathbb{R}_+).$$

We denote by $\tilde{\sigma}$ and σ the cone topologies on $\tilde{C}$ and $\{0\} \times \mathbb{R}_+$ respectively. Then $\tilde{\sigma}$ is p-complete by assumption while σ is the Sorgenfrey topology and hence is p-complete as well. Consequently, the space $\tilde{C} \times (\{0\} \times \mathbb{R}_+)$ is p-complete with the product topology $\tilde{\sigma} \times \sigma$. Now let $u \colon \tilde{C} \times (\{0\} \times \mathbb{R}_+) \to K$ denote the difference mapping $(\tilde{x}, \tilde{y}) \to \tilde{x} - \tilde{y}$, and let $\varkappa$ denote the image of the topology $\tilde{\sigma} \times \sigma$ under u on the cone K. More precisely, $\varkappa$ is obtained by taking as a base of neighbourhoods of $\tilde{z} \in K$ with respect to $\varkappa$ the sets

$$\tilde{z} + u(\tilde{V} \times \tilde{W}) = \tilde{z} + (\tilde{V} - \tilde{W}),$$

where $\tilde{V}$ varies over the $\tilde{\sigma}$-neighbourhoods of $(0, 0)$ in $\tilde{C}$ and $\tilde{W}$ varies over the σ-neighbourhoods of $(0, 0)$ in $\{0\}\times\mathbb{R}_+$. This actually defines a topology on K, and it follows from its definition that u maps $\tilde{C}\times(\{0\}\times\mathbb{R}_+)$ continuously and open onto K with $\varkappa$. Therefore $\varkappa$ is again a p-complete topology.

We wish to prove that every nonempty $\varkappa$-open set U in K is dense in an open subset of $E\times\mathbb{R}$, i.e., that int $\bar{U}\neq\emptyset$ in $E\times\mathbb{R}$. This means that the identity mapping $i:(K,\varkappa)\to E\times\mathbb{R}$ is nearly feebly open. Now observe that by construction the topology $\varkappa$ is invariant under the translations of the form $\tilde{x}\to\tilde{x}+\tilde{y}$, where $\tilde{y}\in K$, which means that for any fixed $\tilde{y}\in K$ the translation $\tilde{x}\to\tilde{x}+\tilde{y}$ maps $(K,\varkappa)$ homeomorphically onto its open subspace $(K+\tilde{y},\varkappa)$. Therefore, in order to prove that i is nearly feebly open, it suffices to check int $\bar{U}\neq\emptyset$ in $E\times\mathbb{R}$ for $\varkappa$-neighbourhoods U of $(0, 0)$ only.

Typically a neighbourhood U of $(0, 0)$ in $(K,\varkappa)$ is of the form

$$U=[(V\times[-1,1])\cap\tilde{C}]-[\{0\}\times[0,1)],$$

where V is a neighbourhood of 0 in E. Now it is easy to see that this implies $W\times[0,1)\subset\bar{U}$, where W is a neighbourhood of 0 contained in $V\cap\bar{C}$. Hence int $\bar{U}\neq\emptyset$ as claimed.

The mapping $i:(K,\varkappa)\to E\times\mathbb{R}$ is continuous and nearly feebly open. K being dense in $E\times\mathbb{R}$, Theorem 1.2 implies that K is residual in $E\times\mathbb{R}$. Consequently, the same is true for $F=K\cap(-K)$. But note that F is a linear subspace of $E\times\mathbb{R}$, and this implies $F=E\times\mathbb{R}$. Indeed, this may be concluded either by [9, Theorem 4(b)] or using the difference theorem from [5]. In the second case one argues as follows. Since F is a second category subset of $E\times\mathbb{R}$ having the Baire property, the difference theorem tells that $F-F$ is a neighbourhood of $(0, 0)$ in $E\times\mathbb{R}$. But $F-F=F$.

Clearly $F=E\times\mathbb{R}$ gives $K=E\times\mathbb{R}$, and this implies $0\in C$ by the definition of K. This ends the proof. □

4. Graph theorem

In this section we present as a first application of our notion of pseudo-completeness the following generalization of a graph theorem of Robinson [12]. First we need a definition.

Let E, F be separated locally convex vector spaces and let Φ be a mapping from E to the set of all nonempty subsets of F. We denote by $G(\Phi)$ the graph of Φ which is

$$\{(x,y)\in E\times F: y\in\Phi(x)\}.$$

Then Φ is called a *convex multifunction* if $G(\Phi)$ is a convex subset of $E\times F$. Various examples for such convex multifunctions may be found in [12] and the references given there.

As usual, for a subset M of E we note $\Phi(M)=\bigcup\{\Phi(x): x\in M\}$, and we call $\Phi(E)$ the range of Φ, noted $R(\Phi)$.

Theorem 4.1. *Let E, F be Banach spaces and let Φ be a convex multifunction from E to F whose graph $G(\Phi)$ is strictly pseudo-complete. Let y be an interior point of $R(\Phi)$. Then Φ is open at every point $x \in \Phi^{-1}(y)$, i.e., given any open ball B_E with centre 0 in E, there exists an open ball B_F with centre 0 in F satisfying*

$$\Phi(x+B_E) \supset y+B_F.$$

Proof. Let $y \in \operatorname{int} R(\Phi)$ and some x having $y \in \Phi(x)$ be fixed. To simplify things, we may assume that $x=0$, $y=0$. Indeed, we may replace Φ by the convex multifunction Φ^* defined by $\Phi^*(z)=\Phi(z+x)-y$, then $0 \in \operatorname{int} R(\Phi^*)$, $0 \in \Phi^*(0)$, and if Φ^* can be proved to be open at $0 \in E$ in our sense, then the openness of Φ at x follows.

First let us observe that $\overline{\Phi(B_E)}$ is a neighbourhood of 0 in F, i.e., Φ is nearly open at 0. Indeed, observe that $\overline{\Phi(B_E)}$ is closed convex, and we prove that it is absorbing. Let $y \in F$ be fixed. Since $0 \in \operatorname{int} R(\Phi)$, there exists an open ball B_F with centre 0 in F such that $B_F \subset R(\Phi)$. Choose $\lambda > 0$ having $\lambda y \in B_F$, $\lambda y \in \Phi(x)$ for some x. Using the convexity of $G(\Phi)$ and $(0,0) \in G(\Phi)$, we find that

$$\mu\lambda y \in \Phi(\mu x)$$

for $0<\mu \leq 1$. We choose μ such that $\mu x \in B_E$, then $\mu\lambda y \in \Phi(B_E)$. This proves the claim.

Since F is a Banach space, we deduce that $\overline{\Phi(B_E)}$ is a neighbourhood of 0 in F. It remains to prove that $\Phi(B_E)$ is semi-closed, for then $\Phi(B_E)$ is a neighbourhood of 0 in F. In view of Theorem 3.1 we have to show that $\Phi(B_E)$ is pseudo-complete. Now observe that

$$\Phi(B_E)=p_F((B_E \times F) \cap G(\Phi)),$$

where p_F denotes the projection $E \times F \to F$. This proves that $\Phi(B_E)$ is pseudo-complete, since Lemma 4.2 below tells that $(B_E \times F) \cap G(\Phi)$ is strictly pseudo-complete, while Lemma 4.3 implies that the projection of this set onto the F-coordinate space is again pseudo-complete, the projection onto the E-coordinate space being contained in B_E, which is a bounded set. This ends the proof of Theorem 4.1. □

Lemma 4.2. *Let C, D be strictly pseudo-complete convex sets in a locally convex Fréchet space E. Then $C \cap D$ is again strictly pseudo-complete.*

Proof. Let us first assume that C, D are strictly pseudo-complete convex cones with vertices at 0 in the Fréchet space E. Let σ_C, σ_D denote the corresponding cone topologies, and let σ be the cone topology on $C \cap D$.

Let (φ, T) on (C, σ_C) and (ψ, S) on (D, σ_D) be strict p-complete webs (see Section 0.2). We may assume that (φ, T) and (ψ, S) have the following additional property: For cofinal branches (t_n) in T, (s_n) in S, one has $\lim_{n\to\infty} \operatorname{diam} \varphi(t_n) = \lim_{n\to\infty} \operatorname{diam} \psi(s_n)=0$, where diam refers to the metric diameter in E. Now let R

denote the tree of height $\aleph_0$ consisting of all finite sequences

$$((t_1, s_1), \ldots, (t_n, s_n)),$$

$n \in \mathbb{N}$, $t_1 <_T \cdots <_T t_n$, $s_1 <_S \cdots <_S s_n$, ordered in the natural way, and let χ be defined on R by setting

$$\chi((t_1, s_1), \ldots, (t_n, s_n)) = \varphi(t_n) \cap \psi(s_n).$$

Then by the definition of the topologies $\sigma_C, \sigma_D, \sigma$ the sets $\chi((t_1, s_1), \ldots, (t_n, s_n))$ are σ-open and, moreover, (χ, R) fulfills the conditions required for a strict web on the space $C \cap D$. We prove that (χ, R) is in fact a p-complete web on $C \cap D$. Suppose (t_n), (s_n) are cofinal branches such that $\chi((t_1, s_1), \ldots, (t_n, s_n)) \neq \emptyset$ for all n. This implies $\varphi(t_n) \neq \emptyset$ and $\psi(s_n) \neq \emptyset$ for all n, hence there exist x, y having $x \in \varphi(t_n)$, $y \in \psi(s_n)$ for all n. But notice that $d(x, y) \leq \operatorname{diam} \varphi(t_n) + \operatorname{diam} \psi(s_n) \to 0$ $(n \to \infty)$, hence $x = y \in \chi((t_1, s_1), \ldots, (t_n, s_n))$ for all n. This proves the result in the case of convex cones C, D with vertices 0.

Now let us consider the general case. This follows from the first part of the proof when we observe that $(C \cap D)^\sim = \tilde{C} \cap \tilde{D}$ holds for the cones $\tilde{C}, \tilde{D}$ and $(C \cap D)^\sim$ associated with $C, D, C \cap D$. $\square$

Lemma 4.3. *Let E, F be locally convex Fréchet spaces and let C be a pseudo-complete convex subset of $E \times F$ such that $p_E(C)$ is bounded in E. Then $p_F(C)$ is pseudo-complete.*

Proof. Let p denote the projection operator $E \times F \times \mathbb{R} \to F \times \mathbb{R}$. Then the cones $\tilde{C}$ and $p_F(C)^\sim$ associated with C, $p_F(C)$ are related by

$$p(\tilde{C}) = p_F(C)^\sim.$$

Since, by assumption, $\tilde{C}$ is p-complete in its cone topology $\tilde{\sigma}$, it suffices to prove that $p|\tilde{C}$ is a continuous and open surjection from $(\tilde{C}, \tilde{\sigma})$ onto $p_F(C)^\sim$ endowed with the corresponding cone topology. Since the continuity of p is clear from the definitions, we are left to check the claimed openness. Clearly it suffices to prove this at the origin $(0, 0, 0)$ in $\tilde{C}$. So let $\tilde{U}$ be a neighbourhood of $(0, 0, 0)$ in $E \times F \times \mathbb{R}$. We may assume that $\tilde{U}$ is of the form

$$\tilde{U} = U_E \times U_F \times [-1, 1]$$

for neighbourhoods U_E, U_F of 0 in E, F. Since $p_E(C)$ is bounded, we find $\lambda \geq 1$ such that $p_E(C) \subset \lambda U_E$. Now $O := (1/\lambda)(U_F \times [-1, 1]) \cap p_F(C)^\sim$ is a neighbourhood of $(0, 0)$ in $p_F(C)^\sim$ with its cone topology. It remains to prove that O is contained in $p(\tilde{U} \cap \tilde{C})$, for this gives the claimed openness of $p|\tilde{C}$ at $(0, 0, 0)$.

Let $y \in U_F$, $|\mu| \leq 1$, $(y, \mu) \in p_F(C)^\sim = p(\tilde{C})$. The case $\mu = 0$ is clear, so let $\mu \neq 0$. There exists x having $(x, y, \mu) \in \tilde{C}$, so

$$\frac{1}{\mu}(x, y) \in C$$

by the definition of the cone $\tilde{C}$, hence $(1/\mu)x \in \lambda U_E$, $x \in \mu\lambda U_E$. This proves $(1/\lambda)(x, y, \mu) \in U_E \times U_F \times [-1, 1] = \tilde{U}$ as desired. $\square$

Remarks 4.4. (1) Theorem 4.1 was obtained in [12] for closed graph convex multifunctions. Therefore our result contains as special cases the Banach open mapping theorem as well as its extension to the CS-closed case. We just mention that various other graph type theorems may be generalized to the pseudo-complete case using similar methods. See for instance [17] for a related graph theorem of a general nature which is adapted to such treatment.

(2) Notice that the boundedness of $p_E(C)$ cannot be omitted in Lemma 4.3. Consider the following example. Let $C \subset l^2(\mathbb{N})$ be the Hilbert cube, then C is closed and hence $\tilde{C}$ is a CS-closed hence strictly pseudo-complete cone in $l^2(\mathbb{N}) \times \mathbb{R}$, but its projection $p(\tilde{C})$ onto the $l^2(\mathbb{N})$-coordinate is no longer pseudo-complete in view of the fact that it is dense in $l^2(\mathbb{N})$ but does not coincide with $l^2(\mathbb{N})$ (cf. Theorem 3.1).

5. Separation of convex sets

The classical separation theorem for convex sets works in the case where one of the sets C, D under consideration has nonempty interior. If this is not the case, separation is not always possible, even for disjoint bounded closed convex sets C, D in a Banach space. Nevertheless, a classical result of Tuckey's (see [6, 14]) tells that separation is possible in this situation when the additional requirement is made that $C - D$ is dense in an open set. Here we obtain a generalization of this result for the case of pseudo-complete convex sets C, D, using a completely different approach.

Theorem 5.1. *Let C, D be disjoint strictly pseudo-complete convex sets in a Banach space E. Suppose that C is bounded and $C - D$ is dense in an open set, i.e.,* $\operatorname{int}(\overline{C-D}) \neq \emptyset$. *Then C, D can be separated by a closed hyperplane.*

Proof. It suffices to prove that $\operatorname{int}(C-D) \neq \emptyset$, for then $0 \notin C - D$ provides a closed hyperplane separating 0 from $C - D$, and this clearly permits separating C and D.

Let $x \in \operatorname{int}(\overline{C-D})$ and let B be a closed ball with centre 0 having $x + B \subset \overline{C-D}$. C being bounded, we find $n \in \mathbb{N}$ such that $C \subset nB$ and, in addition, $x \in nB$. This implies

$$x + B \subset \overline{C - (D \cap (2n+2)B)}.$$

Indeed, let $z \in B$, then there exist (c_k) in C, (d_k) in D having $c_k - d_k \to x + z$ $(k \to \infty)$. But c_k, $x \in nB$, $z \in B$ gives

$$d_k \in nB + nB + B + B, \quad k \geq k_0.$$

Let $D^* = D \cap (2n+2)B$; then D^* is strictly pseudo-complete by Lemma 4.2. We claim that $\operatorname{int}(C - D^*) \neq \emptyset$. Let u denote the difference mapping $(x, y) \to x - y$, then we have

$$C - D^* = u(C \times D^*) = p((C \times D^* \times E) \cap G(u)),$$

where p is the projection $(x, y, z) \to z$ and where $G(u)$ is the graph of u. As a consequence of Lemma 4.2 and Theorem 6.1 below, the set $(C \times D^* \times E) \cap G(u)$ is strictly pseudo-complete. Since its projection onto the first two coordinates (x, y) is bounded, we may apply Lemma 4.3. This yields the pseudo-completeness of $C - D^*$. So $C - D^*$ is semi-closed, and Theorem 3.1 finally implies $\mathrm{int}(C - D^*) \neq \emptyset$. □

Remark 5.2. The result does not carry over to locally convex Fréchet spaces, which may be seen by examples in [14] or [6]. A purely locally convex version of the Tuckey-Klee separation theorem has been obtained by Valdivia [15] using the notion of locally complete sets.

6. An infinite two-person game

In this section we give an internal characterization of strict pseudo-completeness and clarify somewhat the interrelation between the notion of pseudo-completeness and the concept of CS-closedness.

We define an infinite two-person game Γ between players I and II on a convex set C in a Banach space E. Player I starts by choosing a point $x_1 \in C$ and some $\lambda_1 > 0$. Then player II continues by choosing $\varepsilon_1 > 0$. Next player I chooses $x_2 \in C$ and $\lambda_2 > 0$ such that $\lambda_2 < \varepsilon_1$ and $\|\lambda_2 x_2\| < \varepsilon_1$. Now player II chooses $\varepsilon_2 > 0$, and player I continues by choosing $x_3 \in C$, $\lambda_3 > 0$ having $\lambda_3 < \varepsilon_2$ and $\|\lambda_3 x_3\| < \varepsilon_2$, etc. Player II wins the game in case $\sum_{n=1}^{\infty} \lambda_n < \infty$ and, in addition

$$\sum_{n=m}^{\infty} \lambda_n x_n \in \left(\sum_{n=m}^{\infty} \lambda_n \right) \cdot C$$

is satisfied for every $m \in \mathbb{N}$. In all other cases, player I wins.

First we consider the version of the game Γ where both players have complete information, i.e., when choosing their move, they know of all the previous moves.

Theorem 6.1. *Let C be a convex set in a Banach space E. C is strictly pseudo-complete if and only if player* II *has a winning strategy in the game Γ (played with perfect memory).*

Proof. (1) First suppose that player II has a winning strategy Ξ in the game Γ. We have to prove that $\tilde{C}$ is strictly p-complete in its cone topology.

Let T denote the tree of height $\aleph_0$ consisting of all finite sequences $(\tilde{x}_1, \ldots, \tilde{x}_n)$ of elements $\tilde{x}_i$ of $\tilde{C}$, ordered in the natural way. Let us define open sets $\varphi(\tilde{x}_1, \ldots, \tilde{x}_n)$ with respect to the cone topology using induction.

Let $\tilde{x} = (\lambda x, \lambda) \in \tilde{C}$ be fixed. Then $\Xi(x, \lambda) = \varepsilon > 0$. Now define $\varphi(\tilde{x}) = \tilde{x} + (B_\varepsilon \cap \tilde{C})$, where $B_\varepsilon = \{(y, \mu): \|y\| < \varepsilon, |\mu| < \varepsilon\}$. Next let $\tilde{x}_1 = (\lambda_1 x_1, \lambda_1)$, $\tilde{x}_2 = (\lambda_2 x_2, \lambda_2)$ in $\tilde{C}$ be fixed. Suppose that $\varphi(\tilde{x}_1) = \tilde{x}_1 + (B_{\varepsilon_1} \cap \tilde{C})$. If $\tilde{x}_2$ is not contained in $B_{\varepsilon_1} \cap \tilde{C}$, then let $\varphi(\tilde{x}_1, \tilde{x}_2) = \emptyset$. Suppose $\tilde{x}_2 \in B_{\varepsilon_1} \cap \tilde{C}$, then we may apply Ξ and this provides some $\varepsilon_2 = \Xi((x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2)) > 0$. Now choose $\delta_2 > 0$ such that $\delta_2 \leq \varepsilon_2$ and $\tilde{x}_2 + (\bar{B}_{\delta_2} \cap \tilde{C}) \subset B_{\varepsilon_1} \cap \tilde{C}$, so that

$$\varphi(\tilde{x}_1, \tilde{x}_2) := \tilde{x}_1 + \tilde{x}_2 + (B_{\delta_2} \cap \tilde{C})$$

is contained in $\varphi(\tilde{x}_1)$. Proceeding in this way provides us with a pair (φ, T) satisfying conditions (i) and (ii) from Section 0.1 in the strict version. We finish the first part of the proof by showing that condition (p) from Section 0.2 is as well satisfied for (φ, T).

Let $\tilde{x}_i = (\lambda_i x_i, \lambda_i) \in \tilde{C}$, $i = 1, 2, \ldots$ be given with $\varphi(\tilde{x}_1, \ldots, \tilde{x}_i) \neq \emptyset$ for all i. Then by the definition of φ there exist sequences (ε_i) and (δ_i) of positive scalars satisfying

$$\varphi(\tilde{x}_1, \ldots, \tilde{x}_n) = \sum_{i=1}^{n} \tilde{x}_i + (B_{\delta_n} \cap \tilde{C}), \quad \delta_1 = \varepsilon_1,$$

$\delta_n \leq \varepsilon_n$, $\tilde{x}_n + (\bar{B}_{\delta_n} \cap \tilde{C}) \subset B_{\delta_{n-1}} \cap \tilde{C}$, $\varepsilon_n = \Xi((x_1, \lambda_1), \varepsilon_1, \ldots, (x_n, \lambda_n))$. We define a strategy Θ for player I in the game Γ by setting $\Theta(\emptyset) = (x_1, \lambda_1)$, $\Theta((x_1, \lambda_1), \varepsilon_1) = (x_2, \lambda_2)$, $\Theta((x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2), \varepsilon_2) = (x_3, \lambda_3)$, etc. Then the sequence $(x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2), \varepsilon_2, \ldots$ turns out to be the game of I playing with Θ against II playing with Ξ. Since by assumption Ξ is winning, we deduce that

$$\sum_{n=m}^{\infty} \lambda_n x_n \in \left(\sum_{n=m}^{\infty} \lambda_n \right) \cdot C$$

holds for every m. In particular, this implies

$$\tilde{x} = \left(\sum_{n=1}^{\infty} \lambda_n x_n, \sum_{n=1}^{\infty} \lambda_n \right) \in \tilde{C},$$

and it remains to prove that $\tilde{x} \in \varphi(\tilde{x}_1, \ldots, \tilde{x}_m)$ for every m. So let m be fixed. Then we have

$$\begin{aligned} \tilde{x} &= \left(\sum_{n=1}^{m} \lambda_n x_n, \sum_{n=1}^{m} \lambda_n \right) + \left(\sum_{n=m+1}^{\infty} \lambda_n x_n, \sum_{n=m+1}^{\infty} \lambda_n \right) \\ &= \sum_{n=1}^{m} \tilde{x}_n + \left(\sum_{n=m+1}^{\infty} \lambda_n x_n, \sum_{n=m+1}^{\infty} \lambda_n \right) \in \sum_{n=1}^{m} \tilde{x}_n + C. \end{aligned}$$

But note that for every $p > m+1$, $\sum_{n=m+1}^{p} \tilde{x}_n \in \tilde{x}_{m+1} + (B_{\delta_{m+1}} \cap \tilde{C})$ holds, so that $\sum_{n=m+1}^{\infty} \tilde{x}_n \in \tilde{x}_{m+1} + (\bar{B}_{\delta_{m+1}} \cap \tilde{C}) \subset B_{\delta_m} \cap \tilde{C}$ is obtained as a consequence of the definition of the sequence (δ_n). This ends the first part of the proof.

(2) Suppose now that $\tilde{C}$ is strictly p-complete when endowed with its cone topology, and let (φ, T) be a web in accordance with Section 0.2. We define a strategy Ξ for player II in the game Γ.

For $x_1 \in C$, $\lambda_1 > 0$ fixed choose $t_1 \in T$ such that $(\lambda_1 x_1, \lambda_1) \in \varphi(t_1)$ and then choose $\varepsilon_1 > 0$ such that $(\lambda_1 x_1, \lambda_1) + (B_{\varepsilon_1} \cap \tilde{C})$ is contained in $\varphi(t_1)$. Let $\Xi(x_1, \lambda_1) = \varepsilon_1$.

Next suppose that $x_1, x_2 \in C$ and $\lambda_1, \lambda_2 > 0$ are given with $\Xi(x_1, \lambda_1) = \varepsilon_1$ and $\|\lambda_2 x_2\| < \varepsilon_1$, $\lambda_2 < \varepsilon_1$. We have to define $\Xi((x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2))$. Since $(\lambda_2 x_2, \lambda_2) \in B_{\varepsilon_1}$, axiom (ii) of a strict web gives us $t_2 \in T$ having $t_1 <_T t_2$ such that

$$(\lambda_1 x_1, \lambda_1) + (\lambda_2 x_2, \lambda_2) \in \varphi(t_2) \subset (\lambda_1 x_1, \lambda_1) + (B_{\varepsilon_1} \cap \tilde{C}). \tag{*}$$

Now we choose $\varepsilon_2 > 0$ with $\varepsilon_2 \leq \frac{1}{2}\varepsilon_1$ such that

$$(\lambda_1 x_1, \lambda_1) + (\lambda_2 x_2, \lambda_2) + (B_{\varepsilon_2} \cap \tilde{C}) \subset \varphi(t_2)$$

and define $\Xi((x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2)) = \varepsilon_2$.

Proceeding in this way we obtain a strategy Ξ for player II in the game Γ. We prove that Ξ is winning. So let Θ be any strategy for player I and let

$$(x_1, \lambda_1), \varepsilon_1, (x_2, \lambda_2), \varepsilon_2, \ldots \tag{**}$$

represent the game of I with Θ against II with Ξ. Clearly then the series $\sum_{n=1}^{\infty} \lambda_n$ converges in view of $\lambda_n < \varepsilon_{n-1}$ and $\varepsilon_n \leq \frac{1}{2}\varepsilon_{n-1}$, and $\sum_{n=1}^{\infty} \lambda_n x_n$ converges in E in view of $\|\lambda_n x_n\| < \varepsilon_{n-1}$ and the fact that E is complete. It remains to prove that for every m, $\sum_{n=m}^{\infty} \lambda_n x_n$ lies in $(\sum_{n=m}^{\infty} \lambda_n) \cdot C$.

Observe that (φ, T) fulfills condition (p) of Section 0.2 hence there exists $\tilde{x} = (\lambda x, \lambda) \in \bigcap \{\varphi(t_n) : n \geq 1\}$, where (t_n) is the cofinal branch in T corresponding with the sequence (**) using the construction (*). Clearly this implies $\tilde{x} \in \sum_{i=1}^{n} (\lambda_i x_i, \lambda_i) + B_{\varepsilon_n}$ for every n, hence we obtain

$$\tilde{x} = \left(\sum_{n=1}^{\infty} \lambda_n x_n, \sum_{n=1}^{\infty} \lambda_n \right) \in \tilde{C}.$$

Moreover, we have

$$\tilde{x} = \sum_{i=1}^{m} (\lambda_i x_i, \lambda_i) + \sum_{i=m+1}^{\infty} (\lambda_i x_i, \lambda_i) \in \sum_{i=1}^{m} (\lambda_i x_i, \lambda_i) + (B_{\varepsilon_m} \cap \tilde{C}),$$

which yields

$$\sum_{i=m+1}^{\infty} (\lambda_i x_i, \lambda_i) \in \tilde{C},$$

and this gives the desired relation $\sum_{i=m+1}^{\infty} \lambda_i x_i \in (\sum_{i=m+1}^{\infty} \lambda_i) \cdot C$. □

Remark 6.2. Only for convenience did we establish Theorem 6.1 for Banach spaces. Replacing the norm $\| \ \|$ by some invariant metric would give the result for locally convex Fréchet spaces.

A strategy Ξ for player II in the game Γ is called *reasonable* if for any strategy Θ for player I, the sequence (ε_n) resulting from the moves executed by Ξ is summable. Clearly every strategy Ξ for II destined to be winning must a priori be chosen reasonable in this sense. Notice that a reasonable strategy for player II may be defined without a complete knowledge of all the previous moves of both players. In fact, if player II wishes to play in a reasonable way, it suffices for him to know either the last move of his opponent or his own previous move.

Proposition 6.3. *Let C be a convex set in the Banach space E. Then C is CS-closed if and only if every reasonable strategy Ξ for player* II *is automatically winning.*

The proof is immediate from the definition c CS-closedness. But now we may derive Propositions 2.2, 2.5 by combining Proposition 6.3 with Theorem 6.1.

7. Concluding remarks

We conclude our paper by listing some questions and problems. First of all it would be desirable to have an explicit example of a (strictly) pseudo-complete convex set which is not CS-closed. Clearly Theorem 6.1 and Proposition 6.3 suggest the existence of such an example. Also a pseudo-complete set which is not strictly pseudo-complete should be constructed.

It seems likely that Lemma 4.2 is no longer valid if the sets C, D are assumed pseudo-complete only. Clearly, an example of two pseudo-complete sets C, D whose intersection $C \cap D$ is no longer pseudo-complete would in particular provide an example of a pseudo-complete set which is not strictly pseudo-complete.

It is clear from Proposition 6.3 that CS-closedness implies the existence of a winning strategy for player II in the game Γ which takes into account only the last move of the opponent. It would be interesting to know whether, conversely, the existence of such a winning strategy for player II characterizes CS-closedness.

Acknowledgement

We would like to thank the referee for his useful comments and for pointing out an inaccuracy in the original proof of Theorem 6.1.

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